GWYNETH MORELAND'S PROJECT PROPOSAL

1. Introduction

In algebraic geometry, we often seek to understand a variety by understanding its subvarieties. The simplest subvarieties to study are generally curves and divisors– i.e. subvarieties of dimension or codimension 1. But there has been much recent interest in studying subvarieties of intermediate dimension and studying *intermediate cycles*: that is, studying subvarieties of X of dimension $\neq 1, \dim X - 1$, and studying groups generated by these subvarieties modulo some equivalence relation, such as Chow groups, numerical groups, and homology groups.

Intermediate cycles have far-ranging consequences and applications. The Hodge conjecture asks if every Hodge class is a sum of algebraic cycles of intermediate dimension. That is, the Hodge conjecture is fundamentally a question about understanding the intermediate cycles on algebraic varieties. However, intermediate cycles are traditionally very hard to understand, making any work that provides concrete information about their structure useful.

There has been recent work determining certain geometrically-significant *positive cones* of intermediate dimension, namely the effective and nef cones. Effective and nef cones are classically studied in the divisor and curves case due to their birational geometry applications and the tractability of such problems. Recent work (including [?DERV11], [?CC15], [?CLO16], [?RS21], [?S22], [?GL23]) has been done in the intermediate case, but there are still far fewer examples of fully computed nef and effective cones than in the curves and divisor case. A large part of this is due to the difficulty of working with intermediate cycles: the ambient vector spaces they live in are much larger, it is much harder to tell when these cones are finitely generated, and various tools available to us in the curves and divisors setting no longer work. For example, a nef divisor necessarily needs to be pseudoeffective, but this is not true for nef cycles of intermediate dimension.

In my PhD thesis, I provide explicit computations of intermediate nef and effective cones for the case of the Hilbert scheme of three points in $\mathbb{P}^3$. *A priori* these cones do not have to be finitely generated, but I am able to show they are, and furthermore I provide a geometric description for each of the generators.

I am currently a first year postdoc at UIC, and I plan to use my experience with intermediate cycles to help me tackle a variety of questions in the area. My project proposal details the main questions I want to investigate in regards to intermediate cycles: in Section 2, I detail my past accomplishments and some natural followup directions; in Section 3, I outline some projects relating to Lehmann's notion of generalized Iitaka dimension; in Section 4, I propose some questions relating to the connectivity of Hilbert schemes over Grassmannians.

2. Past accomplishments

In algebraic geometry, we are often interested in spaces that parametrize other objects. One such object are Hilbert schemes, which parametrize sufficiently "similar" subvarieties of an algebraic variety X, and help us study families of subvarieties. One class of Hilbert scheme of particular interest are the Hilbert schemes of points, i.e. Hilbert schemes parametrizing 0-dimensional schemes in some variety. Hilbert schemes of points are important for various reasons: namely, they are some of the first Hilbert schemes one encounters and serves as a litmus test for our knowledge on the topic, and their pathologies propagate to other Hilbert schemes. Therefore, understanding the foundational case of Hilbert schemes of points is important to understanding the overall geometry of varieties.

Let $\mathrm{Hilb}_m \mathbb{P}^n$ denote the Hilbert scheme parametrizing 0-dimensional length m subschemes of $\mathbb{P}^n$. These schemes quickly become poorly-behaved for $n \geq 3$: they need not be smooth (e.g. $\mathrm{Hilb}_4 \mathbb{P}^3$), nor irreducible ([?I72],[?CEVV09]), nor reduced (e.g. [?S21]). In my previous work I studied $\mathrm{Hilb}_3 \mathbb{P}^3$, the Hilbert scheme parametrizing collections of three points in $\mathbb{P}^3$ and all limits of such configurations. This space is handleable (it is smooth and irreducible) while still displaying the interesting features of higher dimension (large homology groups, interesting collinear schemes). It is on the "boundary" of reasonable Hilbert schemes of points– that is, incrementing the number of points by even one to consider $\mathrm{Hilb}_4 \mathbb{P}^3$ yields a nonsmooth scheme.

Inspired by Ryan and Stathis's work on $\mathrm{Hilb}_3 \mathbb{P}^2$ [?RS21], and utilizing Rossello-Llompart's results [?RL90] on the cohomology of $\mathrm{Hilb}_3 \mathbb{P}^3$, I proved the following.

Theorem 2.1 ([?M23])**.** *Let* $\mathrm{Hilb}_3 \mathbb{P}^3$ *denote the Hilbert scheme of 3 points in* $\mathbb{P}^3$*. We have explicit formulas for the cone of effective surfaces* $\mathrm{Eff}_2(\mathrm{Hilb}_3 \mathbb{P}^3)$ *and cone of effective threefolds* $\mathrm{Eff}_3(\mathrm{Hilb}_3 \mathbb{P}^3)$*. They are generated by 7 rays in a 10 dimensional vector space and 13 rays in a 10 dimensional vector space, respectively. Each generator has an explicit geometric description.*

By taking the dual cones, we have descriptions for $\mathrm{Nef}^2(\mathrm{Hilb}_3 \mathbb{P}^3)$*, the cone of nef 7-folds, and* $\mathrm{Nef}^3(\mathrm{Hilb}_3 \mathbb{P}^3)$*, the cone of nef 6-folds.*

Consequently, a homology class in $\mathrm{H}_4(\mathrm{Hilb}_3 \mathbb{P}^3)$ *or* $\mathrm{H}_6(\mathrm{Hilb}_3 \mathbb{P}^3)$ *is represented by an algebraic surface or threefold, respectively, if it lies in an explicitly computed, finitely generated cone.*

The proof utilizes a consequence of Kleiman's transversality theorem to convert the problem into largely a study of the $\mathrm{PGL}_4(\mathbb{C})$ orbits of $\mathrm{Hilb}_3 \mathbb{P}^3$. Crucially, I extend the basis for the homology groups of $\mathrm{Hilb}_3 \mathbb{P}^2$ described in [?MS90] to $\mathrm{Hilb}_3 \mathbb{P}^3$ so that this lemma can be applied. In my paper [?M23] I produce this basis and compute the change of basis between it and the one described in [?RL90] using some careful intersection computations. Then using various decomposition, specialization, and Schubert calculus techniques I carry out the orbit analysis needed to prove Theorem 2.1.

There are many natural followup questions one can ask.

Question 2.2. *The dimension of* $H_4(\mathrm{Hilb}_3 \mathbb{P}^n) \cong N_2(\mathrm{Hilb}_3 \mathbb{P}^3)$*, the ambient vector space in which* $\mathrm{Eff}_2(\mathrm{Hilb}_3 \mathbb{P}^n)$ *lives, stabilizes for* $n \geq 4$*. What is the cone* $\mathrm{Eff}_2(\mathrm{Hilb}_3 \mathbb{P}^4)$*, and do these same generators yield* $\mathrm{Eff}_2(\mathrm{Hilb}_3 \mathbb{P}^n)$ *for all* $n \geq 4$*?*

The techniques of my paper suggest that these cones should indeed stabilize. This would yield a large class of examples of effective cones in intermediate dimension. The main barriers are finding the right analog of the basis in [?MS90] and managing the fact that we have less intersection data for $\mathrm{Hilb}_3 \mathbb{P}^4$. Both of these are tractable with the $\mathrm{Hilb}_3 \mathbb{P}^3$ case completed, and I plan to resolve this question during my time at UIC.

Question 2.3. *What is* $\mathrm{Eff}_2(\mathrm{Hilb}_4 \mathbb{P}^2)$*? Does the presence of infinitely many* $\mathrm{PGL}_4(\mathbb{C})$ *orbits lead to a significantly more complex effective cone?*

[?M23] and [?RS21] rely on the PGL action on $\mathrm{Hilb}_3 \mathbb{P}^3$ having finitely many orbits. $\mathrm{Hilb}_4 \mathbb{P}^2$ has infinitely many orbits, but is one of the most well-behaved examples of infinitely many orbits, since it has a $\mathbb{P}^1$-family of orbits that should behave relatively similarly. Solving this question would provide notable progress in the toolkit of studying effective cones in intermediate dimension, and is something I have been in discussion with Tim Ryan of [?RS21] about.

3. Generalized Iitaka Dimension

Iitaka dimension is classically defined for a divisor D in an algebraic variety X, and is an asymptotic measure of how many sections mD has for m large. Lehmann extended this notion to subvarieties of codimension not equal to 1 in [?L19].

If $N_k(X)$ denotes formal sums of subvarieties of X modulo numerical equivalence, and $[Y]$ is the class of a subvariety of X, then generalized Iitaka dimension is an asymptotic measure of how many general points a subvariety equivalent to $m[Y]$ can be made to pass through. Like in the divisor case, the geometry of a map contracting $[Y]$ can be related to the generalized Iitaka dimension. Lehmann poses the following question:

Conjecture 3.1. *Let X be a projective variety and $[Y]$ the class of a subvariety of codimension $k > 1$. Is the Iitaka dimension of $[Y]$ an integer?*

Conjecture 3.1 being true would mean that subvarieties of intermediate dimension, despite being far less well-behaved than divisors, still have some regularity in regards to positivity and interpolation. Currently, the main examples we have for generalized Iitaka dimension are Schubert varieties in the Grassmannians $\mathrm{Gr}(2, m)$. Computing more examples would provide support (or disproof of) this conjecture and give some geometric insight to the classes studied.

These questions also naturally lead to interesting interpolation computations. Interpolation has been a area of rich study ([?LV21], [?CM11], [?C06]) and can tell us a lot about the geometry of a variety. Results in this area tend to be about curves on varieties, so results on intermediate-dimensional subvarieties would be interesting.

Question 3.2. *Do subvarieties of $\mathrm{Gr}(k, n)$ or of other flag varieties have integer Iitaka dimension?*

This is the question I plan to focus the most on, both due to being the natural next step after the $\mathrm{Gr}(2, n)$ result, and because understanding interpolation on Grassmannians is inherently a rich topic. For flag varieties I would start with a more specific and handleable case, like those parametrizing a pair of a line and a point. It would be interesting to see how the methods at the end of [?L19] extend for describing varieties equivalent to $n\Sigma$, where n is an integer and Σ is a Schubert cycle.

The next question would utilize my experience with Hilbert schemes.

Question 3.3. *Do subvarieties of $\mathrm{Hilb}_3 \mathbb{P}^2$ have integer Iitaka dimension? Do subvarieties of $\mathrm{Hilb}_3 \mathbb{P}^3$ have integer Iitaka dimension?*

Since we understand the subvarieties of $\mathrm{Hilb}_3 \mathbb{P}^n$ well and expect them to have reasonable equivalence classes and deformations, it is natural to apply this question to this setting. Specifically, the relevant subvarieties tend to be defined in terms of certain incidence and collinearity conditions, and we expect these conditions to deform nicely.

4. Connectivity of Grassmannians

While Hilbert schemes of points can be extremely poorly-behaved, they are at least connected. This is not always true if our Hilbert schemes parametrize other subvarieties, though. One motivating paper is Seong's paper [?S20] on connectivity of the Hilbert scheme of degree d planar curves in $\mathrm{Gr}(k, n)$, where he shows the Hilbert scheme has two connected components, even though the schemes parametrized all lie in one homology class. The proof relies on the fact that, while the curves are all in one homology class, the planes containing them must lie in one of two homology classes.

Question 4.1. *What Hilbert schemes over Grassmannians have multiple components?*

One should be able to generate a few examples of Hilbert schemes over Grassmannians with multiple components by using a similar idea: look at schemes who are contained in some intermediate type of scheme that must also be contained in the Grasmmannian, and see if that forces multiple components. The question remains of: are there examples that don't fall into this category? Such examples would give us new insights into the geometry of Grassmannians.

There are also interesting questions regarding the connectivity of Hilbert schemes of points on Grassmannians. One usually shows $\mathrm{Hilb}_m \mathbb{P}^n$ is connected by showing you can get from any point to one corresponding to a monomial ideal, and then from a monomial ideal to a distinguished point, corresponding to a *lexicographic ideal*. It would be interesting to investigate if connected Hilbert schemes over Grassmannians have some nice analog of this distinguished point, and what the corresponding subscheme of $\mathrm{Gr}(k,n)$ should look like, potentially using some of the machinery of [?SS23].

5. Career Development

The extra time granted to me by the reduced teaching or potentially an extra year of postdoctoral studies at UIC would greatly help my research. I would have more time for the proposed research questions, and more time for traveling to collaborate and share my mathematical ideas. The NSF MSPRF would greatly benefit my career and put me in a more competitive position when I next enter the academic job market.

6. Justification for Sponsoring Scientist and Institution

Izzet Coskun is an expert on intermediate cycles, having authored multiple papers on the subject ([?CC15], [?CLO16]) and also an expert on Hilbert schemes of points ([?ABCH13]). He has a strong track record with postdoctoral mentorship, with many of his postdocs being going on to be prolific experts in algebraic geometry. Furthermore, UIC has a flourishing research scene, with the algebraic geometry group winning multiple RTG grants over the years. Among this group is Lawrence Ein, who has also worked in intermediate cycles ([?DERV11]). Such an environment will provide a great opportunity for me to expand my skills and grow as a researcher.

7. Broader Impacts

Contributing to mathematics outreach and increasing diversity in math is incredibly important to me. I contributed greatly to the outreach of my department during my graduate career and look forward to continuing this during my postdoc. As a graduate student, I co-organized lunches for Women in Math and Real Representations (a social and support group for underrepresented groups in mathematics) at Harvard for three years. I also served as a mentor for a year for 4 undergraduates as part of Math Includes, a department diversity and belonging initiative. In between graduate school and my postdoc beginning, I served as a course assistant to the undergraduate course of Notre Dame's summer school on rationality and hyperbolicity, where I helped introduce the undergraduates to algebraic geometry, guided them through problem sessions, and answered questions about graduate school.

As a current first year postdoc at UIC, I plan to contribute to mathematics outreach by helping with UIC's Undergraduate Math Symposium, a weekend undergraduate research conference, and attending the department lunches organized by the DEI committee, which provide an opportunity for undergraduate students to meet faculty and learn about the department. test